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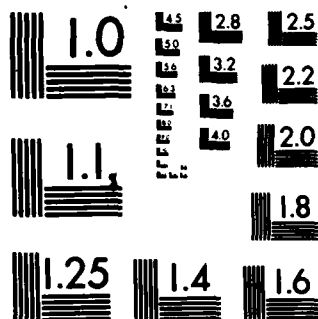
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A NON-CLUSTERING PROPERTY OF STATIONARY SEQUENCES

by

Arif Zaman

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Key Words and Phrases: Clustering, Stationary Sequences, Cyclic Sums

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A Non-clustering Property of Stationary Sequences

by

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Abstract

For a random sequence of events, with indicator variables X_i , the behavior of the expectation $E\left(\frac{X_k + \dots + X_{k+m-1}}{X_1 + \dots + X_n}\right)$ for $1 \leq k \leq k+m-1 \leq n$ can be taken as a measure of clustering of the events. When the measure on the X 's is i.i.d., or even exchangeable, a symmetry argument shows that the expectation can be no more than m/n . When the X 's are constrained only to be a stationary sequence, the bound deteriorates, and depends on k as well. When m/n is small, the bound is roughly $2m/n$ for k near $n/2$ and is like $(m/n) \log n$ for k near 1 or n . The proof given is partly constructive, so these bounds are nearly achieved, even though there is room for improvement for other values of k .

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1. Introduction.

In considering portions of larger, but still finite strings of random variables, the following problem arose. If X_1, \dots, X_n is part of a stationary sequence of zeros and ones, one would not expect the ones within that portion to clump together, intuitively because each X_i is as likely as any other to have the value one. Based on that intuitive argument, one could expect the expression $\sup_{P \in S} E_P \left\{ \frac{X_k + \dots + X_{k+m-1}}{X_1 + \dots + X_n} \right\}$ (note: $0/0 = 0$) where $1 \leq k \leq k+m-1 \leq n$, and S is the set of stationary probability measures on binary sequences, to behave roughly like m/n . Indeed, if the probability P is restricted to be i.i.d. or even exchangeable, a simple symmetry argument yields a supremum of m/n , achieved when the X_i are identically 1. For the case of stationarity, the upper bounds on the supremum for m/n small are roughly $2m/n$ when k is near $n/2$, and like $(m/n) \log n$ for k closer to 1 or n (thm. 7). The key result is a constructive proof that finds the P which achieves the supremum for the two cases of $m = 1, k = 1$, and $m = 1, k = (n+1)/2$ (thm. 2).

I would like to thank Professor Michael Steele for insisting that this could be done, and Professor Larry Shepp for an improvement in the proof. I would also like to acknowledge the many simplifications and improvements suggested by the referee.

2. Results.

We shall immediately narrow our concern to the simpler problem of finding bounds for

$$R_{k,n} = \sup_{P \in S} E_P \left\{ \frac{X_k}{X_1 + \dots + X_n} \right\} \quad \text{for } 1 \leq k \leq n. \quad (1)$$

Notice that the variables X_{n+1}, X_{n+2}, \dots do not appear in the above expression, so only the marginal distribution of (X_1, \dots, X_n) affects the values of $R_{k,n}$. A small amount of notation is needed for the next theorem, which makes use of this observation.

A loop is a finite sequence a_1, \dots, a_m of zeros and ones. Subscripts out of range will be taken circularly, so that $a_0 = a_m$, and $a_{m+1} = a_1$. For a loop a and any positive integer n , the measure $P_{a,n}$ gives mass $1/m$ to each of $(a_1, \dots, a_n), (a_2, \dots, a_{n+1}), \dots, (a_m, \dots, a_{m+n-1})$.

Theorem 1.

If a binary sequence X has a stationary distribution, then the marginal distribution of (X_1, \dots, X_n) can be written as a convex combination of measures $P_{a,n}$ for $a \in A_n$, where A_n is a finite set of loops. Moreover, every $P_{a,n}$ is the marginal of some infinite stationary distribution.

More details, and a proof of this can be found in Zaman (1983) or Hobby and Ylvasaker (1964). Since expectation is a linear functional, thm. 1 allows replacing the maximization over S in eq. 1 by maximization over $P_{a,n}$ for $a \in A_n$, yielding

$$R_{k,n} = \max_{a \in A_n} E_{P_{a,n}} (X_k / \sum_{j=1}^n X_j). \quad (2)$$

Using the definition of $P_{a,n}$, the expectation can be further decomposed into

$$E_{P_{a,n}} (X_k / \sum_{j=1}^n X_j) = \frac{1}{m} \sum_{i=1}^m (a_{i+k} / \sum_{j=1}^n a_{i+j}) \quad (3)$$

where m is the length of the loop a . In a completely unrelated problem, sums of the form given in the right side of eq. 3 have been given the name cyclic sums, e.g. Daykin (1970).

Equations 2 and 3 convert the original probability problem of eq. 1 into a finite maximization of a function over a set of loops. This maximization is performed for chosen values of k in the appendix to prove the following key theorem.

Theorem 2.

(a) When $k = 1$ or n , the maximum in eq. 2 is achieved for $a = 0^{n-1}1^\beta$ (the notation 0^{n-1} refers to a block of $n-1$ zeros) for some number β depending on n .

(b) When $k = (n+1)/2$ for odd n , the maximum in eq. 2 is achieved for $a = 0^{k-1}1$.

Corollary 3.

Define

$$\alpha(n) = \sup_{\beta \geq 1} (n+\beta)^{-1} \sum_{i=1}^{\beta} 1/i. \quad (4)$$

Then,

$$R_{k,n} = \begin{cases} \alpha(n-1) & \text{if } k = 1 \text{ or } n & (a) \\ 2/(n+1) & \text{if } k = (n+1)/2 & (b) \end{cases}$$

The corollary is actually proved as a step in proving thm. 2, but can also be proved by writing out eq. 3 for the loops given in thm. 2.

Using these equalities for $R_{1,n}$ and $R_{(n+1)/2,n}$, a general bound for $R_{k,n}$ is easy to get. Theorems 4 and 5 do just that. The bounds of thm. 4 are depicted graphically in fig. 1.

Theorem 4.

Define

$$\alpha(k,n) = \sup_{n-k \leq \beta} (k+\beta)^{-1} [(n-k)/\beta + \sum_{i=n-k}^{\beta-1} 1/i].$$

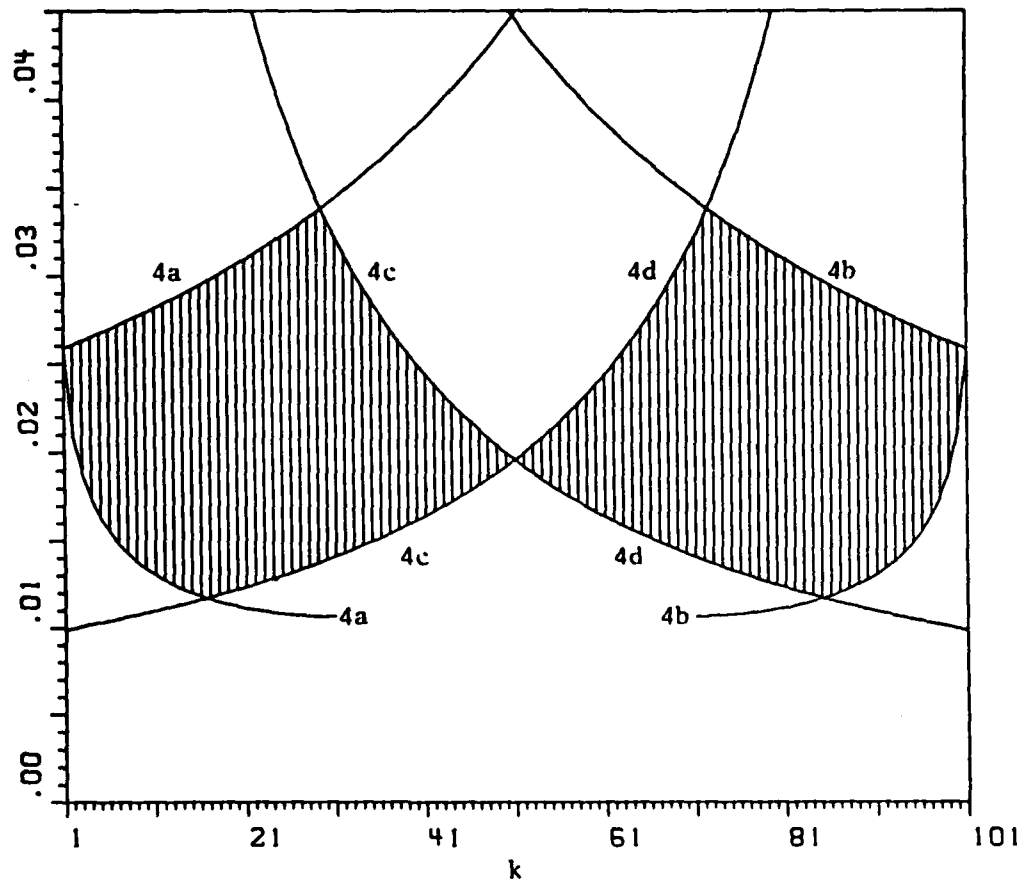


Fig. 1: Bounds on $R_{k,n}$ as a function of k , for $n = 101$.

The area between the upper and lower bounds of thm. 4 is shaded to indicate the possible region for $R_{k,n}$. The different bounds are labelled by the equation number in thm. 4.

Then

- (a) $\alpha(n-k, n) \leq R_{k,n} \leq \alpha(n-k)$ when $2k-1 \leq n$
- (b) $\alpha(k-1, n) \leq R_{k,n} \leq \alpha(k-1)$ when $2k-1 \geq n$
- (c) $1/(n+1-k) \leq R_{k,n} \leq 1/k$ when $2k-1 \leq n$
- (d) $1/k \leq R_{k,n} \leq 1/(n+1-k)$ when $2k-1 \geq n$.

Proof:

Parts (b) and (d) follow from (a) and (c) respectively, once the symmetry condition

$$R_{k,n} = R_{n-k+1,n} \quad (5)$$

is established. To prove this, note that if $P_{a,n}$ is the distribution of (X_1, \dots, X_n) then the distribution of (X_n, \dots, X_1) is given by $P_{a',n}$ for $a' = (a_n, \dots, a_1)$. Now for any loop a ,

$$E_{P_{a,n}}(X_k / \prod_{j=1}^n X_j) = E_{P_{a',n}}(X_{n+1-k} / \prod_{j=1}^n X_j)$$

from which eq. 5 follows.

The upper bound in (a) follows from Cor. 3a by

$$R_{k,n} \leq \sup_{P \in S} E_P(X_k / \prod_{j=k}^n X_j) = R_{1,n+1-k} = \alpha(n-k).$$

Similarly, for part (c), the result of Cor. 3b shows that for $2k-1 \leq n$

$$R_{k,n} \leq \sup_{P \in S} E_P(X_k / \prod_{j=1}^{2k-1} X_j) = R_{k,2k-1} = 1/k.$$

The lower bounds have been included in the theorem to get some idea on the room for improvement of these bounds. It is conjectured that the actual values of $R_{k,n}$ are much closer to the lower bounds than to the upper bounds. The lower bound (a) is obtained by using eq. 3 to get for $k \leq (n+1)/2$

$$R_{k,n} \geq \sup_{\substack{a=0^{n-k-1}\beta \\ k \leq \beta \leq n}} E_p \left(X_k / \sum_{j=1}^n X_j \right) \\ \geq \sup_{k \leq \beta \leq n} (n+\beta-k)^{-1} \left[(k-1)/\beta + \sum_{i=k}^{\beta} 1/i \right].$$

The lower bound in (c) is achieved by letting $a = 0^{n-k-1}$. For that value of a , if $2k-1 \leq n$ then by eq. 3

$$E_p \left(X_k / \sum_{j=1}^n X_j \right) = \frac{1}{n+1-k}.$$

It is not difficult to find loops which give even higher lower bounds, but that does not seem to be the more fruitful direction of moving the bounds. \square

Theorem 5.

$$R_{k,n} \leq \frac{1+\log(n-1)}{n} \quad \text{for } n \geq 3.$$

Before giving a proof, a logarithmic approximation for the function α will be established.

Lemma 6.

$$\frac{\log n - \log(\log n) - 1}{n} \leq \alpha(n) \leq \frac{\log n}{n} \quad \text{for } n \geq 3.$$

Proof:

Let β^* be a value of β which achieves the maximum in eq. 4, so that

$$\alpha(n) = (n+\beta^*)^{-1} \sum_{i=1}^{\beta^*} 1/i. \quad (6)$$

A crude bound to the harmonic series in eq. 6 gives

$$\alpha(n) \leq (1+\log \beta^*)/(n+\beta^*). \quad (7)$$

By calculus, the function $(1+\log x)/(n+x)$ for $x \geq 1$ reaches its maximum value of $(\log x^*)/n$ when $x^* \log x^* = n$. If $n > e$, $\log x^*$ can be bounded by

$$\log n - \log \log n \leq \log x^* \leq \log n. \quad (8)$$

Plugging this information about the maximum into eq. 7

$$\alpha(n) \leq (1+\log \beta^*)/(n+\beta^*) \leq (\log x^*)/n \leq (\log n)/n,$$

establishing the second inequality of the lemma.

For the first inequality, let x^* be as before, define $\beta = [x^*]$ (the integer part), and for notational convenience let $\ell = \log n - \log \log n$ which is the term on the left side of eq. 8. Then

$$\begin{aligned} \alpha(n) &\geq (n+\beta)^{-1} \sum_{i=1}^{\beta} 1/i \geq (n+x^*)^{-1} \log x^* \\ &\geq (n+n/\ell)^{-1} \ell = n^{-1} \ell^2 / (1+\ell) = n^{-1} [\ell - 1 + (\ell+1)^{-1}] \\ &> (\ell-1)/n. \end{aligned} \quad (9)$$

The last inequality substitutes a prettier expression at the cost of some precision. \square

The proof of thm. 5 then amounts to the following. By eq. 5

$$\begin{aligned} \max_k R_{k,n} &= \max_{k \leq (n+1)/2} R_{k,n} \\ \text{(by thm. 4a, c)} &\leq \max_{k \leq (n+1)/2} \{(1/k) \wedge \alpha(n-k)\} \\ \text{(by lem. 6)} &\leq \max_{k \leq (n+1)/2} \{(1/k) \wedge \log(n-k)/(n-k)\} \end{aligned} \quad (10)$$

Since $1/k$ is decreasing and the second function increasing as k increases, the maximum in eq. 10 is attained at some $k = k^*$ for which the two functions are equal. Thus

$$\begin{aligned}\max_k R_{k,n} &= 1/k^* = \log(n-k^*)/(n-k^*) \\ &= [1 + \log(n-k^*)]/n,\end{aligned}$$

where the last expression follows by some algebra. Since $k^* \geq 1$, replacing it by 1 gives the claimed result in thm. 5. \square

Returning to the original problem as stated in the introduction, one can state the following theorem based only on the definition of $R_{k,n}$.

Theorem 7.

$$\sup_{P \in S} E_P \left\{ \frac{\sum_{j=k}^{k+m-1} X_j}{\sum_{j=1}^n X_j} \right\} \leq \sum_{j=k}^{k+m-1} R_{j,n}.$$

For example, this proves that for any stationary measure P ,

$$E_P \left\{ \frac{X_k + \dots + X_{k+m-1}}{X_1 + \dots + X_n} \right\} \leq (m/n) [1 + \log(n-1)]$$

and for blocks near the middle

$$E_P \left\{ \frac{X_{-k} + \dots + X_k}{X_{-n} + \dots + X_n} \right\} \leq \frac{1}{n+1} + 2 \log\left(\frac{n}{n-k}\right) \leq (2k+1)/(n-k)$$

by using the values of $R_{k,n}$ given in theorems 5 and 4c, d.

APPENDIX

Proof of Theorem 2a.

The appendix will use eqns. 2, 3, 4, 5 and lem. 6 from the previous section. It is to be noted that these do not use thm. 2 in any way and are mainly definitional equations. To avoid repeating awkward summations, for the loop $a = a_1, \dots, a_m$ we define

$$S(j, k) = \sum_{i=j}^k a_i$$

$$S_i = S(i-n+1, i)$$

$$T_i = a_i / S_i$$

$$T(j, k) = \sum_{i=j}^k T_i.$$

By eq. 5, $R_{1,n} = R_{n,n}$. We will choose to work with $R_{n,n}$ for which eq. 3 can be written as

$$E_{p_{a,n}} (X_n / \sum_{j=1}^n X_j) = T(1, m) / m. \quad (A.1)$$

Consider the case when a is of the special form $0^{n-1}1^x$ for some integer x . Working out the sums involved in eq. A.1, for this a

$$E_{p_{a,n}} (X_n / \sum_{j=1}^n X_j) = (n-1+x)^{-1} \sum_{i=1}^x 1/i \leq \alpha(n-1). \quad (A.2)$$

It is easy to see that in eq. A.2 equality is achieved for some value of $x \leq n$ which we shall denote by $\beta(n-1)$ (the argument $n-1$ will be assumed from now on). The proof that amongst the set of all loops, the given loop $0^{n-1}1^\beta$ maximizes the expectation will be done by contradiction. Assume there is some

$a = a_1, \dots, a_m$ and $\epsilon > 0$, for which

$$T(1, m)/m > \alpha(n-1) + \epsilon. \quad (\text{A.3})$$

The method of proof involves a stepwise modification of a . At each step the previous sequence will be denoted by a , and the modified one by a' . The variables m' , for the length of a' , as well as S' and T' will similarly be defined for a' . After each step, for the modified sequence the inequality

$$T'(1, m')/m' > \alpha(n-1) \quad (\text{A.4})$$

will be proved. Yet after a finite number of steps, the sequence a' will essentially look like $0^{n-1}1^\beta$, providing the contradiction.

Step 1.

Let m' be a multiple of m , large enough so that $n/m' < \epsilon$, for the ϵ in eq. A.3, and also $m' > 5n$ (this last restriction is not necessary, but allows the treatment of a loop as a long open string).

We have $a = a_1, \dots, a_m$.

Let $a' = 0^{n-1}a_n, \dots, a_{m'}$.

To prove eq. A.4 note that $a'_i \leq a_i$, so $S'_i \leq S_i$. So for $i = n, \dots, m'$ we have $T'_i \geq T_i$, and for $i = 1, \dots, n-1$, $T'_i \leq 1$. Hence

$$T(1, m') \leq (n-1) + T'(n, m').$$

Since m' is a multiple of m ,

$$\begin{aligned} \alpha(n-1) + \epsilon &< T(1, m)/m \\ &= T(1, m')/m' \\ &\leq [(n-1) + T'(1, m')]/m' \\ &\leq \epsilon + T'(1, m')/m' \end{aligned}$$

which proves eq. A.4.

Step 2.

Now $a = 0^{n-1}a_n, a_{n+1}, \dots, a_m$. Define $b = S(n, 2n-1)$.

Let $a' = 0^{n-1}1^b 0^{n-b}a_{2n}, \dots, a_m$.

Note that a' is simply a , with the block a_n, \dots, a_{2n-1} rearranged so that all of its b ones are to the left of its zeros. We pause to prove the following lemma about switching the order of a neighboring pair of 0 and 1.

Lemma 8.

Let a and a' be two loops of the same length m , identical except that $a_{n+j} = a'_{n+j+1} = 0$ and $a'_{n+j} = a_{n+j+1} = 1$. If $a_{j+1} = 0$, then

$$T(1, m) \leq T'(1, m).$$

Proof:

The proof consists simply of noting that the only difference between T_i and T'_i is $T_{2n+j} \leq T'_{2n+j}$, $T_{n+j} = T'_{n+j+1}$ and $T'_{n+j+1} = T_{n+j}$. \square

Applying lem. 8 repeatedly over a large block yields

Corollary 9.

If a has a block of zeros $a_{j+1} = \dots = a_{j+b} = 0$ then construct a' by rearranging the block $a_{n+j}, \dots, a_{n+j+b}$ so that the ones are to the left of the zeros, but otherwise a and a' are identical. Then the conclusion of lem. 8 is still valid.

Returning to step 2 in the construction, $\alpha(n-1) < T(1, m)/m \leq T'(1, m')/m'$, where the first inequality was established in step 1, the second follows from cor. 9.

Step 3.

Now $a = 0^{n-1}1^b 0^{n-b}a_{2n}, \dots, a_m$.

Let $a' = 0^{n-1}1^\beta 0^{n-\beta}a_{2n}, \dots, a_m$

so that $m' = m + \beta - b$.

By the definition of β in eq. A.2,

$$T(1, n+b-1) = \sum_{i=1}^b 1/i \leq (n+b-1)\alpha(n-1) \quad (\text{A.5})$$

$$T'(1, n+\beta-1) = \sum_{i=1}^{\beta} 1/i = (n+\beta-1)\alpha(n-1).$$

For the remaining values $i = n+b, \dots, m$ we have $T_i = T'_{i+\beta-b}$ if $\beta \geq b-1$.

When $\beta < b-1$ the only difference is that $S_i > S'_{i+\beta-b}$ for $i = 2n, \dots, 2n+b-\beta-2$, so that in all cases

$$T(n+b, m) \leq T'(n+\beta, m'). \quad (\text{A.6})$$

Combining eqns. A.5 and A.6

$$T(1, m) - T'(1, m') \leq (b-\beta)\alpha(n-1).$$

This implies eq. A.4 as can be seen by this simple lemma.

Lemma 10.

If $T(1, m) - T(1, m') \leq (m-m')\alpha$ and $T(1, m)/m > \alpha$ then $T'(1, m')/m' > \alpha$.

Proof:

$$0 < T(1, m) - m\alpha < T'(1, m') - m'\alpha. \quad \square$$

Step 4.

If $b > \beta$, return to step 2; otherwise $n-b \geq n-\beta$, so the second block of zeros in a has at least $n-\beta$ elements. Let a_c be the first occurrence of a 1 in $a_{2n+\beta-1}, \dots, a_m$.

Now $a = 0^{n-1}10^{n-\beta}a_{2n}, \dots, a_m$.

Let $a' = 0^{n-1}10^{n-1}a_c, \dots, a_m$,

so that $m' = m+2n+\beta-c-1$.

Note that $T(1, 2n-1) = T'(1, 2n+\beta-2)$, $T'_{2n+\beta-1} = 1$ and $T(c+1, m) = T'(2n+\beta, m')$ so that

$$T(1, m) - T'(1, m') \leq T(2n, 2n+\beta-2) + T_c - 1. \quad (A.7)$$

Let $d = S(2n, 2n+\beta-2)$ so that there are $n-d-1$ zeros in $a_n, \dots, a_{2n+\beta-2}$. Then each S_i for $i = 2n, \dots, 2n+\beta-2$ sums at most $n-d-1$ zeros, and at least $d+1$ ones, i.e., each $S_i \geq d+1$. Since a_i and hence T_i is nonzero d times for $i = 2n, \dots, 2n+\beta-2$

$$T(2n, 2n+\beta-2) \leq d/(d+1). \quad (A.8)$$

We will separate out three cases, and in each case establish

$$T(1, m) - T'(1, m') \leq (m-m')\alpha(n-1), \quad (A.9)$$

which would imply eq. A.4 by lemma 10.

Case 1: $2n+\beta-1 \leq c < 3n$.

Here $(m-m') \geq 0$ and $d = S_k - 1$, so eqns. A.7 and A.8 imply

$$T(1, m) - T'(1, m') \leq (S_k - 1)/S_k + 1/S_k - 1 = 0,$$

establishing eq. A.9.

Case 2: $c \geq 3n$ and $n \neq 4, 6, 8$ or 10 .

Since $d \leq \beta-1$ and $m-m' \geq n+1-\beta$, using eqns. A.7, A.8, we need to show

$$(\beta-1)/\beta \leq (n+1-\beta)\alpha(n-1). \quad (A.10)$$

to prove eqn. A.9. Looking at table 1, this holds for all given values of n except 4, 6, 8, 10. For values beyond the table, eq. A.7 was checked numerically up to $n = 100$, and the logarithmic approximations of lemma 6 will be used after that. Since β maximizes eq. A.2, we have

Table 1:

<u>n</u>	<u>$\beta(n-1)$</u>	<u>$\alpha(n-1)$</u>
1	1	1.000000
2	1	.500000
3	2	.375000
4	3	.305556
5	3	.261905
6	4	.231481
7	4	.208333
8	5	.190278
9	5	.175641
10	6	.163333
11	6	.153125
12	6	.144118
13	7	.136466
14	7	.129643
15	8	.123539

$$\alpha(n-1) \geq [n-1+(\beta-1)]^{-1} \sum_{i=1}^{\beta-1} 1/i$$

$$= \left(\frac{n+\beta-1}{n+\beta-2} \right) \alpha(n-1) - \left(\frac{1}{n+\beta-2} \right) (1/\beta)$$

which gives $\alpha(n-1) \leq 1/\beta$. Since $\beta\alpha(n-1) \leq 1$ and $(\beta-1)/\beta \leq 1$,

$$(n+1-\beta)\alpha(n-1) - (\beta-1)/\beta \geq (n+1)\alpha(n-1) - 2$$

$$\text{(by lem. 6)} \quad \geq (n+1) \frac{\log(n-1) - \log \log(n-1) - 1}{n-1} - 2$$

$$\geq 0 \quad \text{for } n \geq 87.$$

The final inequality can be calculated for $n = 87$, and since the penultimate expression is an increasing function of n , all larger n must also satisfy it. But this establishes eq. A.10 and hence A.9 for all $n \neq 4, 6, 8$ or 10 .

Case 3: $c \geq 3n$ and $n = 4, 6, 8$, or 10 .

This case is further broken into three subcases each involving a verification by table 1.

(3a) If $c = 3n$ and $S_c > 1$ then $T_c \leq 1/2$ so if

$$(\beta-1)/\beta - 1/2 \leq (n+1-\beta)\alpha(n-1)$$

then eq. A.9 is satisfied.

(3b) If $c = 3n$ and $S_c = 1$ then $S(2n+1, 2n+\beta-2) = 0$, and so

$T(2n, 2n+\beta-2) = T_{2n} \leq 1/\beta$. Using this in eq. A.7, we need to verify

$$1/\beta \leq (n+1-\beta)\alpha(n-1).$$

(3c) If $c > 3n$ then $m-m' \geq n+2-\beta$ and we need

$$(\beta-1)/\beta \leq (n+2-\beta)\alpha(n-1).$$

As these cases are exhaustive, and in each case eq. A.4 is true, Step 4 is complete.

Step 5.

Now $a = 0^{n-1}1^\beta 0^{n-1}a_{2n+\beta-1}, \dots, a_m$

Let $a' = 0^{n-1}a_{2n+\beta-1}, \dots, a_m 0^{n-1}1^\beta$.

Since a' is just a rotation of a , $T(1, m) = T'(1, m')$, so eq. A.4 will hold. Now, return to step 2 unless

$$a = 0^{n-1}1^\beta 0^{n-1}1^\beta \dots 0^{n-1}1^\beta. \quad (\text{A.11})$$

At every return to step 2, some elements of the original sequence are deleted or reordered into blocks of $0^{n-1}1^\beta$. Since no new disordered elements are created at any step, the procedure must stop after a finite number of steps. Since at each step eq. A.4 was verified, for the final a of eq. A.11 we must have

$$T(1, m)/m > \alpha(n-1)$$

yet simply computing,

$$T(1, m)/m = (n-1+\beta)^{-1} \sum_{i=1}^{\beta} 1/i = \alpha(n-1)$$

providing the contradiction which proves the theorem. \square

Proof of Theorem 2b.

Let n be odd, $k = (n+1)/2$, and $a = a_1, \dots, a_m$. As notation, define

$$S(j, j') = \sum_{i=j}^{j'} a_i$$

$$T_i = a_{i+k}/S(i+1, i+n)$$

$$T(j, j') = \sum_{i=j}^{j'} T_i$$

so that eq. 3 can be written as

$$E_{p_{a,n}} (X_k / \sum_{j=1}^n X_j) = T(1, m)/m.$$

For any loop a ,

$$T(1, k) = \sum_{i=1}^k a_{i+k}/S(i+1, i+n) \leq \sum_{i=1}^k a_{i+k}/S(k+1, n+1) = 1.$$

As this holds for all loops, it will also hold for the loop

$(a_{hk+1}, a_{hk+2}, \dots, a_{hk+n})$ for any integer h . Thus

$$T(hk+1, (h+1)k) \leq 1 \quad \text{for } h = 0, 1, 2, \dots$$

Adding these up for $h = 0, 1, \dots, m-1$,

$$m > \sum_{h=0}^{m-1} T(hk+1, (h+1)k) = T(1, mk) = kT(1, m), \quad (A.12)$$

because a is periodic with period m . Rewriting A.12 gives

$$T(1, m)/m \leq 1/k = 2/(n+1) \quad (\text{A.13})$$

for any loop a . On the other hand, it is straightforward to verify that the loop $a = 0^{k-1}1$ achieves the upper bound in eq. A.13, thus proving thm. 2b and cor. 3b simultaneously. \square

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→ For a random sequence of events, with indicator variables X_i , the behavior of the expectation $E[(X_k + \dots + X_{k+m-1}) / (X_1 + \dots + X_n)]$ for $1 \leq k \leq k+m-1 \leq n$ can be taken as a measure of clustering of the events. When the measure on the X 's is i.i.d., or even exchangeable, a symmetry argument shows that the expectation can be no more than m/n . When the X 's are constrained only to be a stationary sequence, the bound deteriorates, and depends on k as well. When m/n is small, the bound is roughly $2m/n$ for k near $n/2$ and is like $(m/n) \log n$ for k near 1 or n . The proof given is partly constructive, so these bounds are nearly achieved, even though there is room for improvement for other values of k .

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